A Note on the Hardness of Graph Diameter Augmentation Problems

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Abstract

A graph has *diameter* D if every pair of vertices are connected by a path of at most D edges. The Diameter-D Augmentation problem asks how to add the a number of edges to a graph in order to make the resulting graph have diameter D. It was previously known that this problem is NP-hard [2], even in the $D = 2$ case. In this note, we give a simpler reduction to arrive at this fact and show that this problem is W[2]-hard.

Keywords: Graph augmentation, graph diameter, algorithms, fixed-parameter tractability, W[2]-hard, domination, reduction

1 Introduction

A graph G has *diameter* D if every pair of vertices are connected by a path of at most D. The Graph Diameter-D Augmentation problem takes as input a graph $G = (V, E)$ and a value k and asks whether there exists a set E_2 of new edges so that the graph $G_2 = (V, E \cup E_2)$ has diameter D. This problem was known to be NP-hard for $D \geq 3$ [6] and was later shown to remain hard for the $D = 2$ case [3]. The proof in [3] reduced a restricted (but still NP-hard [2]) 3-Sat problem to a relaxed dominating set problem (which they called Semi-Dominating Set) which was then reduced to Diameter-2 Augmentation. In this note, we provide a reduction to Diameter-2 Augmentation directly from Dominating Set, which not only provides a cleaner proof of NP-hardness but also establishes that Diameter-2 Augmentation is W[2]-hard.

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An algorithm is called *fixed-parameter tractable* (or FPT) if its runtime is $O(f(k)n^c)$ where n is the input size, f is a function of k which does not depend on n and c is a constant. When the value k is fixed, this is essentially a polynomial runtime, and in particular for any fixed k it is the same polynomial (up to coefficients.) FPT algorithms have received much attention lately as many NP-hard problems have been shown to be fixed-parameter tractable. For instance, the Vertex Cover problem has an algorithm ([1]) running in $O(1.2738^k + kn)$ which is linear in n for any fixed k. Analogous to the idea of NP-hardness, there is a measure of hardness for parameterized problems which depend on parameterized reductions. Some well-known parameterized-hard problems are Clique (which is W[1]-hard) and Dominating Set (which is W[2]-hard). These results and a thorough introduction to parameterized problems can be found in [5]. Being parameterized-hard also has implications for the approximatibility of the problem: namely, a problem which is W[1]-hard is unlikely to have an efficient polynomial-time approximation scheme (EPTAS) [4].

1.1 The Reduction

We proceed with a reduction from the parameterized dominating set problem to the parameterized diameter-2 augmentation problem after a formal description of each of these problems and of what constitutes a parameterized reduction. In this report, we consider input graphs which are connected.

Problem 1. Dominating Set
Input: A graph $G = (V, E)$ and a positive integer k.
Task: To determine if there exists a set $S \subseteq V$ of size at most k such that for every $v \in V \setminus S$ there is some $s \in S$ where $\{s, v\}$ is an edge.

Problem 2. Diameter-2 Augmentation
Input: A graph $G = (V, E)$ and a positive integer k.
Task: To determine if there exists a set of at most k edges that can be added to G so that the resulting graph has diameter 2.

We must reduce Dominating Set to Diameter-2 Augmentation via a *parameterized reduction*. That is, we must give a mapping that sends a yes-instance (G_1, k_1) of Dominating Set to a yes-instance (G_2, k_2) of Diameter-2 Augmentation where k_2 depends on k_1 alone. We will provide a mapping here where $k_2 = k_1$.

Let (G_1, k_1) be an instance of Dominating Set, where $G_1 = (V_1, E_1)$. We construct a graph G_2 with two copies of G_1 called U_1 and U_2. Any two

vertices $u_1 \in U_1$ and $u_2 \in U_2$ that correspond to the same vertex $v \in V_1$ will be called *twins*. For each vertex w in U_1, join an edge between w and its twin in U_2. Let w_i and w_j be any two distinct vertices in $U_1 \cup U_2$. In G_2, create a new set Y of vertices $y(w_1, w_2)$ such that Y induces a complete graph and each vertex $y(w_1, w_2)$ is adjacent to w_1 and to w_2. Finally, we create in G_2 a vertex z adjacent to every vertex of Y and adjacent to no vertex in $U_1 \cup U_2$, and create a vertex x adjacent to z alone.

Note that G_2 has diameter at most 3. Every pair of vertices in G_2 which is not connected by a 2-path must be x with some $w_i \in U_1 \cup U2$. It is easy to see that if a dominating set D of G_1 contained k vertices, then the set of edges $\{x, d\}, d \in D$ forms a diameter-2 augmenting set (also of size k) for G_2. We now prove the converse.

Theorem 1. *G_1 has a dominating set of size k if and only if $G_2 = (V_2, E_2)$ has an augmenting set of edges S such that $H = (V_2, E_2 \cup S)$ has diameter 2.*

Proof. Given a k-augmenting set of G_2, we will construct a dominating set D of G_1 also of size k. If an augmenting set of G_2 only contains edges from x to vertices in U_1 we will call it *proper*. We can extract a dominating set of U_1 (and thus of G_1) from a proper diameter-2 augmenting set S of G_2 simply by taking all the vertices of U_1 that are adjacent to x in S.

Say that S is a solution set of edges from Diameter-2 Augmentation on input G_2. We will show how to construct a proper augmenting set from S of at most the same size as S. For any vertex $w \in U_1 \cup U_2$, there must be a 2-path (or less) joining x to w. If such a 2-path ever passing through the vertex z, we can remove the $\{z, w\}$ edge from S and add $\{x, w\}$ to S instead. Note that such an edge-swap can never increase the diameter of the graph. We will provide a sequence of edge-swapping rules to the set S until we arrive at a proper augmenting set.

Rule 1. If S has an edge $\{z, w\}$ for any $w \in G_2$ then remove $\{z, w\}$ and add $\{x, w\}$.

To describe the rest of the rules, we partition $U_1 \cup U_2$ into the following sets:

i) U_x = vertices u in $U_1 \cup U_2$ such that $\{x, u\} \in S$

ii) U^- = vertices u in $U_1 \cup U_2$ that are not in U_x and there is an edge $\{x, y(u, w)\} \in S$

iii) U^+ = vertices in $U_1 \cup U_2$ that are not in $U_x \cup U^-$

Clearly, these three sets are disjoint from each other and their union is exactly $U_1 \cup U_2$. To arrive at a proper augmenting set, the edges of S joining vertex x to the set Y will have to be removed. It should be easy to verify that each of the following rules will not increase the diameter of H.

Rule 2. If S has an edge $\{x, y(a, b)\}$ with a adjacent to b then remove $\{x, y(a, b)\}$ and add the edge $\{x, a\}$.

Rule 3. If S has an edge $\{x, y(a, b)\}$ with a in U_x then remove $\{x, y(a, b)\}$ and add the edge $\{x, b\}$.

Rule 4. If S has edge $\{x, y(a, b)\}$ and a is adjacent to some c in U_x then remove $\{x, y(a, b)\}$ and add the edge $\{x, b\}$.

Rule 5. If S has two edges $\{x, y(a, b)\}$ and $\{x, y(b, c)\}$ then remove both of them and add the edges $\{x, y(a, c)\}$ and $\{x, b\}$.

Rule 6. If S has two edges $\{x, y(a, b)\}$ and $\{x, y(c, d)\}$ such that a is adjacent to c in G_2 then remove $\{x, y(a, b)\}$ and $\{x, y(c, d)\}$ and add $\{x, a\}$ and $\{x, a(b, d)\}$.

After applying Rules 3-6 we may have to return to Rule 2 and repeat this process, if any such edges would exist. Each rule reduces the number of edges from x to the Y set, so this process must indeed terminate.

Once we arrive at a point where none of the above rules can be applied any further, we make the following observations:

Proposition 1. The set U^- is empty.

Proof. If any edge exists in U^- then Rule 6 could be applied, so we have that U^- is a stable set. If any edge existed from U^- to U_x then this would imply Rule 4 could be applied. Now consider any vertex u in U^-: it must have an adjacent twin vertex, call it u^t, and it must be in U^+. Every vertex in U^+ must have a 2-path to x, but U^+ are the vertices which are not adjacent to any vertex in Y, and so every U^+ must be adjacent to one neighbour of x in U_x. Now if u^t is adjacent to some $a \in U_x$ then so is u, which violates Rule 4. Hence no such u can exist, so U^- is empty once these rules can no longer be applied. $\square$

Proposition 1 tells us that all edges in the augmenting set S must be from x to U_x. We introduce one last rule to make this augmenting set proper:

Rule 7. If S has any $\{x, u\}$ edge where $u \in U_2$ then let u^t be the twin of u and remove $\{x, u\}$ and add the edge $\{x, u^t\}$.

Now with a proper augmenting set, we can extract a dominating set of size at most k in U_1. In the above notation, this is exactly the set U_x when there are no more edge-swap rules that can be applied.

□

1.2 The Diameter-Improvement Problem

Consider the following problem, which asks if the diameter of a graph can be improved (i.e. lowered):

Problem 3. DIAMETER IMPROVEMENT
INPUT: A graph $G = (V, E)$ and a positive integer k.
TASK: To determine if there exists a set of at most k edges that can be added to G so that the resulting graph has a smaller diameter than G.

As previously noted, the graph resulting from the reduction from DOMINATING SET to DIAMETER-2 AUGMENTATION had diameter 3 from its construction. Finding an augmenting edge set that improves this graph to diameter 2 will in fact solve the dominating set problem on the original (pre-reduction) graph. This provides a proof that DIAMETER IMPROVEMENT is itself W[2]-hard (and NP-complete,) even when restricted to input graphs of diameter 3.

2 Concluding Remarks

We gave a reduction to DIAMETER-2 AUGMENTATION directly from DOMINATING SET which establishes the fixed-parameter hardness of DIAMETER-2 AUGMENTATION with respect to the augmenting set size. This also provides a proof of NP-completeness for DIAMETER-2 AUGMENTATION which reduced directly from a known and standard NP-complete problem. We identified the DIAMETER IMPROVEMENT and noted that it is fixed-parameter hard. Future considerations include finding exact exponential-time algorithms that are faster than brute-force searching for DIAMETER-2 AUGMENTATION, as well as the classification of subclasses of graphs for which DIAMETER-2 AUGMENTATION or DIAMETER IMPROVEMENT can be solved in polynomial time.

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